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A Characterization of Box $\frac{1}{d}$-Integral Binary Clutters*

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Let Q_6 denote the port of the dual Fano matroid F_7^* and let Q_7 denote the clutter consisting of the circuits of the Fano matroid F_7 that contain a given element. Let $\mathscr{L}$ be a binary clutter on E and let $d \geqslant 2$ be an integer. We prove that all the vertices of the polytope $\{x \in \mathbb{R}_+^E \mid x(C) \geqslant 1 \text{ for } C \in \mathscr{L}\} \cap \{x \mid a \leqslant x \leqslant b\}$ are $\frac{1}{d}$-integral, for any $\frac{1}{d}$-integral a, b, if and only if $\mathscr{L}$ does not have Q_6 or Q_7 as a minor. This includes the class of ports of regular matroids. Applications to graphs are presented, extending a result from Laurent and Poljak [7].

1. THE MAIN RESULT

Let $\mathscr{L}$ be a collection of subsets of a set E. $\mathscr{L}$ is called a *clutter* if, for all $A, B \in \mathscr{L}$, $A = B$ whenever $A \subseteq B$. Given an integer $d \geqslant 1$, a vector is $\frac{1}{d}$-*integral* if all its components belong to $\frac{1}{d}\mathbb{Z} := \{\frac{i}{d} \mid i \in \mathbb{Z}\}$.

DEFINITION 1.1. *Let $\mathscr{L}$ be a clutter on E. We say that $\mathscr{L}$ is box $\frac{1}{d}$-integral if $\mathscr{L} = \{\varnothing\}$ or, for all $a, b \in (\frac{1}{d}\mathbb{Z})^E$, each vertex of the polyhedron*

$$Q(\mathscr{L}, a, b) := \{x \in \mathbb{R}_+^E \mid x(C) \geqslant 1 \text{ for } C \in \mathscr{L},\ a_e \leqslant x_e \leqslant b_e \text{ for } e \in E\}$$

is $\frac{1}{d}$-integral. Equivalently, $\mathscr{L}$ is box $\frac{1}{d}$-integral if, for all subsets $I \subseteq E$ and all $a \in (\frac{1}{d}\mathbb{Z})^I$, each vertex of the polyhedron

$$Q(\mathscr{L}, a) := \{x \in \mathbb{R}_+^E \mid x(C) \geqslant 1 \text{ for } C \in \mathscr{L},\ x_e = a_e \text{ for } e \in I\}$$

is $\frac{1}{d}$-integral.

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We shall mostly use the second definition for box $\frac{1}{d}$-integral clutters.

Given a clutter $\mathscr{L}$ on E and a subset Z of E, set $\mathscr{L}\backslash Z = \{A \in \mathscr{L} \mid A \cap Z = \varnothing\}$ and let $\mathscr{L}/Z$ consist of the minimal members of $\{A - Z \mid A \in \mathscr{L}\}$; both $\mathscr{L}/Z$ and $\mathscr{L}\backslash Z$ are clutters. The operations are called, respectively, *deletion* and *contraction* of Z. A *minor* of $\mathscr{L}$ is obtained from $\mathscr{L}$ by a sequence of deletions and contractions.

Let $\mathscr{M}$ be a matroid on a groundset $E \cup \{l\}$, where l is a distinguished element of the groundset, and let $\mathscr{C}$ denote the family of circuits of $\mathscr{M}$. The *l-port* of $\mathscr{M}$ is the clutter $\{C \mid C \cup \{l\} \in \mathscr{C}\}$. A clutter is *binary* if it is the port of some binary matroid.

The binary clutters Q_6 and Q_7 are defined, respectively, on six and seven elements. Q_6 is the clutter on the set $\{1, 2, 3, 4, 5, 6\}$ consisting of the sets $\{1, 3, 5\}$, $\{1, 2, 6\}$, $\{2, 3, 4\}$, and $\{4, 5, 6\}$. Q_7 is the clutter on the set $\{1, 2, 3, 4, 5, 6, 7\}$ consisting of the sets $\{1, 4, 7\}$, $\{2, 5, 7\}$, $\{3, 6, 7\}$, $\{1, 2, 6, 7\}$, $\{1, 3, 5, 7\}$, $\{2, 3, 4, 7\}$, and $\{4, 5, 6, 7\}$.

The following result is the main result of the paper. Applications to graphs are given in Section 5.

THEOREM 1.2. *Let $\mathscr{L}$ be a binary clutter on a set E, $\mathscr{L} \neq \{\varnothing\}$. The following assertions are equivalent*:

(i) *$\mathscr{L}$ does not contain Q_6 or Q_7 as a minor,*

(ii) *$\mathscr{L}$ is box $\frac{1}{d}$-integral for each integer $d \geqslant 1$,*

(iii) *$\mathscr{L}$ is box $\frac{1}{d}$-integral for some integer $d \geqslant 2$.*

Observe that, for $d = 1$, $\mathscr{L}$ is box $\frac{1}{d}$-integral if and only if $\mathscr{L}$ has the following *weak max-flow-min-cut property* (since the weak max-flow-min-cut property is closed under minors [10]): $\mathscr{L} = \{\varnothing\}$ or, for each $w \in \mathbb{Z}_+^E$, the program

$$\begin{array}{llll} \min & w^T x & & \\ \text{subject to} & x(C) \geqslant 1 & \text{for all} & C \in \mathscr{L} \\ & x_e \geqslant 0 & \text{for all} & e \in E \end{array}$$

has an integer optimizing vector.

A nonempty clutter $\mathscr{L}$ is said to be *Mengerian* if $\mathscr{L} = \{\varnothing\}$, or both the above program and its dual

$$\begin{array}{llll} \max & 1^T y & & \\ \text{subject to} & \sum_{e \in C} y_C \leqslant w_e & \text{for} & e \in E \\ & y_C \geqslant 0 & \text{for} & C \in \mathscr{L} \end{array}$$

have integer optimizing vectors for all $w \in \mathbb{Z}_+^E$. Seymour [10] showed that a clutter $\mathscr{L} \neq \{\emptyset\}$, which is a matroid port, is Mengerian if and only if $\mathscr{L}$ is binary and does not have any Q_6 minor. Therefore, from Theorem 1.2, the class of the binary clutters which are box $\frac{1}{d}$-integral for some integer $d \geqslant 2$ is strictly contained in the class of Mengerian binary clutters.

The characterization of the clutters with the weak max-flow-min-cut property is a hard and unsolved problem, even within the class of matroid ports (see [10], [4]).

Theorem 1.2 does not hold for ports of arbitrary matroids. For this, consider the matroid U_4^2 on four elements whose circuits are the sets $\{1, 2, 3\}$, $\{1, 2, 4\}$, $\{1, 3, 4\}$ and $\{2, 3, 4\}$. (Recall that a matroid is binary if and only if it does not contain U_4^2 as a minor (Tutte [15]).) The 4-port of U_4^2 is the clutter C_3 consisting of the sets $\{1, 2\}$, $\{1, 3\}$ and $\{2, 3\}$. It is easy to check that C_3 is box $\frac{1}{d}$-integral if and only if d is even. Hence, the assertions (ii) and (iii) of Theorem 1.2 are not equivalent for the clutter C_3.

PROPOSITION 1.3. *Let d be an odd integer and let $\mathscr{L}$ be a matroid port. If $\mathscr{L}$ is box $\frac{1}{d}$-inteqral, then $\mathscr{L}$ is a binary clutter.*

Proof. Let $\mathscr{L}$ be the l-port of a matroid $\mathscr{M}$. We can suppose that $\mathscr{M}$ is connected. Assume that $\mathscr{L}$ is box $\frac{1}{d}$-integral. Then $\mathscr{L}$ does not have C_3 as a minor, see Proposition 3.2. Therefore, $\mathscr{M}$ does not have a minor U_4^2 using the element l. This implies that $\mathscr{M}$ does not have any minor U_4^2 (Bixby [3]). Therefore, $\mathscr{M}$ is a binary matroid. Hence, $\mathscr{L}$ is a binary clutter. ∎

In order to prove Theorem 1.2, it suffices to show the implications (iii) ⇒ (i) and (i) ⇒ (ii). The implication (iii) ⇒ (i) is implied by the following facts:

- box $\frac{1}{d}$-integrality is preserved under minors, see Proposition 3.2.
- Q_6 is not box $\frac{1}{d}$-integral, for each integer $d \geqslant 2$, see Proposition 3.3.
- Q_7 is not box $\frac{1}{d}$-integral, for each integer $d \geqslant 2$, see Proposition 3.4.

The most difficult part is to show the implication (i) ⇒ (ii). For this, we use as a main tool a decomposition result for matroids without minor F_7^* using a given element l, stated in Theorem 2.3 (Tseng and Truemper [14], Truemper [12]).

The proof of Theorem 1.2 is presented in Sections 3 and 4. In Section 2, we recall some results about matroids and the decomposition result that we need here. We present in Section 5 some applications of our main result.

We conclude with another, equivalent, definition for box $\frac{1}{d}$-integral clutters, which is related to the "$\mathscr{F}$-property" considered by Nobili and Sassano [8]. Given a clutter $\mathscr{L}$ on E, $\mathscr{L} \neq \{\varnothing\}$, consider the polyhedron

$$Q(\mathscr{L}) := \{x \in \mathbb{R}_+^E \mid x(C) \geqslant 1 \text{ for all } C \in \mathscr{L}\}.$$

Given a k-dimensional face F $(k \geqslant 0)$ of $Q(\mathscr{L})$, a subset $J \subseteq E$ is said to be *basic* for F if there exist $|E| - k$ equations $x(C_i) = 1$ $(C_i \in \mathscr{L}$, for $1 \leqslant i \leqslant |E| - k)$ defining F whose projections on $\mathbb{R}^J$ are linearly independent. Then, one can check that $\mathscr{L}$ is box $\frac{1}{d}$-integral if and only if the following property holds: For each k-dimensional face F of $Q(\mathscr{L})$ $(k \geqslant 0)$ for each basic set $J \subseteq E$ for F and for each $x \in F$, $x_e \in \frac{1}{d}\mathbb{Z}$ for all $e \in J$ whenever $x_e \in \frac{1}{d}\mathbb{Z}$ for all $e \in E - J$. This property corresponds to the "$\mathscr{F}$-property" considered (in blocking terms and in a slightly more general setting) by Nobili and Sassano [8].

2. Preliminaries on Matroids

We recall here several well known results on matroids that we need for the paper. We refer to [17], [13] for details on the material covered in this section.

We use the following notation. Given a set A and elements $a \in A$, $b \notin A$, $A - a$, $A + b$ denote, respectively, $A - \{a\}$ and $A \cup \{b\}$. If x, y are two binary vectors, then $x \oplus y$ denotes the binary vector obtained by taking the componentwise sum of x and y modulo 2.

Representation Matrix

Let $\mathscr{M}$ be a binary matroid on a set E, i.e., there exists a binary matrix M whose columns are indexed by E such that a subset of E is independent in $\mathscr{M}$ if and only if the corresponding subset of columns of M is linearly independent over the field $GF(2)$. Such a matrix M is called a *representation matrix* of $\mathscr{M}$.

Let X be a base of $\mathscr{M}$ and set $Y = E - X$. For $y \in Y$, let C_y denote the fundamental circuit of y in the base X, i.e., C_y is the unique circuit of $\mathscr{M}$ such that $y \in C_y$ and $C_y \subseteq X + y$. Let B denote the $|X| \times |Y|$ matrix whose columns are the incidence of the sets $C_y - y$ for $y \in Y$. Then, the matrix $[I \mid B]$ is a representation matrix of $\mathscr{M}$ and B is called a *partial representation matrix* of $\mathscr{M}$.

For $x \in X$, let Σ_x denote the fundamental cocircuit of x with respect to the base X, i.e., Σ_x is the unique cocircuit of $\mathscr{M}$ such that $x \in \Sigma_x$ and $\Sigma_x \subseteq Y + x$. The row of B indexed by x is the incidence vector of the set $\Sigma_x - x$.

For $y \in Y$ and $x \in C_y$, the set $X' = X - x + y$ is also a base of $\mathscr{M}$. The partial representation matrix B' of $\mathscr{M}$ in the base X' is easily obtained from

B by *pivoting* with respect to the (x, y)-entry of B. Let $R_{x'}$, $x' \in X$, denote the rows of B; they are vectors in $\{0, 1\}^Y$. Pivoting with respect to the (x, y)-entry of B amounts to replacing $R_{x'}$ by $R_{x'} \oplus R_x \oplus (1, 0, ..., 0)$ (where 1 is in the y-position) for each $x' \in C_y$, $x' \neq x, y$.

Let $\mathscr{C}$ denote the family of circuits of $\mathscr{M}$. A set $C \subseteq E$ is called a *cycle* of $\mathscr{M}$ if $C = \varnothing$ or C is a disjoint union of circuits of $\mathscr{M}$. Equivalently, if M is a representation matrix of $\mathscr{M}$, then the cycles are the subsets whose incidence vectors u satisfy $Mu \equiv 0 \pmod 2$.

Minors

Let Z be a subset of E. The matroid $\mathscr{M} \backslash Z$, obtained by *deletion* of Z, is the matroid on $E - Z$ whose family of circuits is $\mathscr{C} \backslash Z$. The matroid $\mathscr{M}/Z$, obtained by *contraction* of Z, is the matroid on $E - Z$ whose circuits are the nonempty sets of $\mathscr{C}/Z$. Note that contracting a loop or coloop is the same as deleting it. A *minor* of $\mathscr{M}$ is obtained by a sequence of deletions and contractions. Every minor of $\mathscr{M}$ is of the form $\mathscr{M} \backslash Z/Z'$ for some disjoint subsets Z, Z' of E. Given $e \in E$, the minor $\mathscr{M} \backslash Z/Z'$ *uses the element* e if $e \notin Z \cup Z'$; in other words, e belongs to the groundset of $\mathscr{M} \backslash Z/Z'$.

Minors can be easily visualized in the partial representation matrix. Let B be the partial representation matrix of $\mathscr{M}$ corresponding to the base X. If $Z \subseteq X$, then the matrix obtained from B by deleting its rows indexed by Z is a partial representation matrix of $\mathscr{M}/Z$. If $Z \subseteq Y$, then the matrix obtained from B by deleting its columns indexed by Z is a partial representation matrix of $\mathscr{M} \backslash Z$.

k-Sum

Let $\mathscr{M}_i$ be a binary matroid on E_i, for $i = 1, 2$. Let $\mathscr{M}$ be the binary matroid on $E = E_1 \triangle E_2$ whose cycles are the subsets of E of the form $C_1 \triangle C_2$, where C_i is a cycle of $\mathscr{M}_i$ for $i = 1, 2$. We consider the cases:

- $E_1 \cap E_2 = \varnothing$, then $\mathscr{M}$ is called the 1-*sum* of $\mathscr{M}_1$ and $\mathscr{M}_2$
- $|E_1|, |E_2| \geqslant 3$, $E_1 \cap E_2 = \{e_0\}$ and e_0 is not a loop or a coloop of $\mathscr{M}_1$ or $\mathscr{M}_2$, then $\mathscr{M}$ is the 2-*sum* of $\mathscr{M}_1$ and $\mathscr{M}_2$.

k-Separation

Let $r(\cdot)$ denote the rank function of the matroid $\mathscr{M}$ on E. Let $k \geqslant 1$ be an integer. A *k-separation* of $\mathscr{M}$ is a partition (E_1, E_2) of the groundset E satisfying

$$\begin{cases} |E_1|, |E_2| \geqslant k, \\ r(E_1) + r(E_2) \leqslant r(E) + k - 1. \end{cases}$$

When $r(E_1) + r(E_2) = r(E) + k - 1$, the separation is called *strict*. The matroid $\mathscr{M}$ is said to be *k-connected* if it has no j-separation for $j \leqslant k - 1$. Throughout the paper, 2-connected will be abbreviated as *connected*.

If $\mathcal{M}$ has a strict *k-separation* (E_1, E_2), then it admits a partial representation matrix of a special form. Indeed, let X_2 be a maximal independent subset of E_2 and let $X_1 \subseteq E_1$ such that $X = X_1 \cup X_2$ is a base of $\mathcal{M}$, so $|X_1| = r(E_1) - k + 1$ and $|X_2| = r(E_2)$. Set $Y_i := E_i - X_i$, for $i = 1, 2$. The partial representation matrix B of $\mathcal{M}$ in the base X has the form shown in Fig. 1. The rank of the matrix D is equal to $k - 1$.

In the case of a strict 1-separation, the matrix D is identically zero. Then, $\mathcal{M}$ is the 1-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$.

In the case of a strict 2-separation, the matrix D has rank 1 and, thus, has the form shown in Fig. 2.

The set $\tilde{Y}_1$ consists of the elements $y \in Y_1$ for which $X_1 + y$ is an independent set of $\mathcal{M}$. So, if $y \in \tilde{Y}_1$, then the fundamental circuit of y in the base X is of the form $\tilde{X}_2 \cup A_y \cup \{y\}$ with $A_y \subseteq X_1$.

Given two elements $e_1 \in \tilde{X}_2$ and $e_2 \in \tilde{Y}_1$, we consider the matroids $\mathcal{M}_1 = \mathcal{M}/(X_2 - e_1)\backslash Y_2$ and $\mathcal{M}_2 = \mathcal{M}/X_1\backslash(Y_1 - e_2)$ defined, respectively, on $E_1 \cup \{e_1\}$ and $E_2 \cup \{e_2\}$. It follows from the next Proposition 2.1 that $\mathcal{M}$ is the 2-sum of $\mathcal{M}_1$ and $\mathcal{M}_2$ (after renaming e_1 as e_0 in $\mathcal{M}_1$ and e_2 as e_0 in $\mathcal{M}_2$). A set $C \subseteq E$ is said to be *crossing* if $C \cap E_1 \neq \emptyset$ and $C \cap E_2 \neq \emptyset$.

PROPOSITION 2.1. (i) *Let C be a circuit of $\mathcal{M}$. Then,*

- *either $C \subseteq E_i$ and C is a circuit of $\mathcal{M}_i$, for some $i \in \{1, 2\}$,*
- *or C is crossing and $(C \cap E_i) + e_i$ is a circuit of $\mathcal{M}_i$, for $i = 1$ and 2. Moreover, $(C \cap E_1) \cup \tilde{X}_2$ and $(C \cap E_2) \triangle \tilde{X}_2$ are circuits of $\mathcal{M}$.*

Every circuit of $\mathcal{M}_i$ arises in one of the two ways indicated above.

(ii) *Let C, C' be two crossing circuits of $\mathcal{M}$. Then, $(C \cap E_i) \triangle (C' \cap E_j)$ is a cycle of $\mathcal{M}$ for any $i, j \in \{1, 2\}$.*

Proof. (ii) follows directly from (i) and (i) is easy to check after observing that, for a circuit C of $\mathcal{M}$, C is crossing if and only if $|C \cap \tilde{Y}_1|$ is odd. ∎

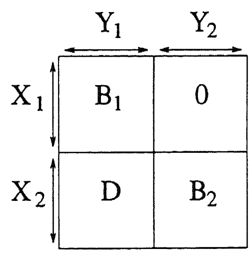

FIGURE 1

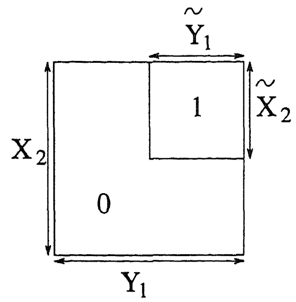

FIGURE 2

In the case of a strict 3-separation, the matrix D has rank 2. Moreover, if $|E_1|, |E_2| \geqslant 4$ and $\mathcal{M}$ is 3-connected, it can be shown that $\mathcal{M}$ has a partial representation matrix B of the form shown in Fig. 3, with $D_{12} = D_2 D_1$ (see [12]).

PROPOSITION 2.2. *Suppose $\mathcal{M}$ has a strict 3-separation (E_1, E_2) with $|E_1|, |E_2| \geqslant 4$ and consider the partial representation matrix of $\mathcal{M}$ from Fig. 3. If $\{y, z, l\}$ is a circuit of the matroid $\mathcal{M}/(X_1 - x)\backslash(Y_1 - \{y, z\})$, then the partition $(E_1, E_2 - l)$ of $E - l$ is a strict 2-separation of the matroid $\mathcal{M}/l$.*

Proof. Let a, b denote the rows of D_1 indexed, respectively, by e, f and let u, v denote the columns of D_2 indexed, respectively, by y, z. So, a, b are vectors indexed by the elements $y' \in Y_1 - \{y, z\}$ and u, v are indexed by the

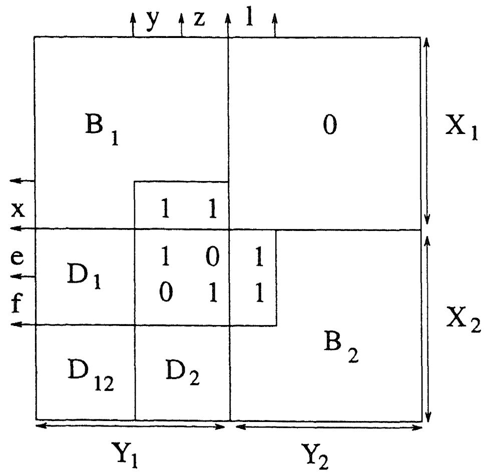

FIGURE 3

elements $x' \in X_2 - \{e, f\}$. Let w denote the vector whose components are the (x', l)-entries, for $x' \in X_2 - \{e, f\}$, of the first column of B_2. Since the set $\{y, z, l\}$ is a circuit of the matroid $\mathcal{M}/(X_1 - x)\backslash(Y_1 - \{y, z\})$, we deduce that $w = u \oplus v$.

The (e, l)-entry of B is equal to 1, hence the set $X' = X - e + l$ is again a base of $\mathcal{M}$. Let B' denote the partial representation matrix of $\mathcal{M}$ in the base X'. So B' can be obtained from B by pivoting with respect to its (e, l)-entry. Pivoting will affect only the rows of B indexed by $X_2 - e$. Let D' denote the submatrix of B' with row index set $X_2 - e + l$ and with column index set Y_1. It is not difficult to check that the row of D' indexed by f is the vector $(a \oplus b, 1, 1)$ and that each other row of D' indexed by some element of $X_2 - \{e, f\}$ is one of the two vectors $(a \oplus b, 1, 1)$ or $(0, ..., 0, 0, 0)$. Therefore, the submatrix of D' with row index set $X_2 - e$ has rank 1. This shows that the partition $(E_1, E_2 - l)$ of $E - l$ is a strict 2-separation of the matroid $\mathcal{M}/l$. ∎

Fano Matroid

The Fano matroid F_7 is the matroid on $\{1, 2, 3, 4, 5, 6, 7\}$ whose circuits are the seven sets $\{1, 2, 3\}$, $\{1, 4, 7\}$, $\{1, 5, 6\}$, $\{2, 4, 6\}$, $\{2, 5, 7\}$, $\{3, 4, 5\}$ and $\{3, 6, 7\}$ (the lines of the Fano plane) together with their complements. The dual *Fano matroid* F_7^* is the dual of F_7, its circuits are $\{4, 5, 6, 7\}$, $\{2, 3, 5, 6\}$, $\{2, 3, 4, 7\}$, $\{1, 3, 5, 7\}$, $\{1, 3, 4, 6\}$, $\{1, 2, 6, 7\}$ and $\{1, 2, 4, 5\}$ (the complements of the lines of the Fano plane).

By symmetry, there is only one port for F_7^*. The 7-port of F_7^* is the clutter Q_6, already defined earlier, consisting of the sets $\{4, 5, 6\}$, $\{2, 3, 4\}$, $\{1, 3, 5\}$ and $\{1, 2, 6\}$.

Observe that every one-element contraction of F_7 has a 2-separation. For example, the sets $\{1, 4\}$ and $\{2, 3, 5, 6\}$ form a strict 2-separation of $F_7/7$.

We also consider the *series-extension* F_7^+ of the Fano matroid F_7, obtained by adding a new element "8" in series with, say, the element "7" i.e., $\{7, 8\}$ is a cocircuit of F_7^+. Hence, F_7^+ is the matroid defined on $\{1, 2, 3, 4, 5, 6, 7, 8\}$ whose circuits are the sets C for which C is a circuit of F_7 with $7 \notin C$, and the sets $C \cup \{8\}$ for which C is a circuit of F_7 with $7 \in C$. Up to symmetry, there are two distinct l-ports of F_7^+, depending whether l is one of the two series elements 7, 8, or not. We denote by Q_7 the l-port of F_7^+ when l is a series element of F_7^+. Then, for $l = 8$, Q_7 consists of the sets $\{1, 4, 7\}$, $\{2, 5, 7\}$, $\{3, 6, 7\}$, $\{1, 2, 6, 7\}$, $\{1, 3, 5, 7\}$, $\{2, 3, 4, 7\}$ and $\{4, 5, 6, 7\}$, i.e., Q_7 consists of the circuits of F_7 containing the point 7.

We use the following facts about regular matroids ([13], [15], [17]). A matroid is *regular* if it does not have any F_7, F_7^*, or U_4^2 minor. Let $\mathcal{M}$ be a regular matroid and let $M = [I \mid B]$ be a binary matrix representing $\mathcal{M}$ over $GF(2)$. Then the 1's of B can be replaced by ± 1's so that the resulting matrix $\tilde{B}$ is totally unimodular, i.e., each square submatrix of $\tilde{B}$ has determinant 0, ± 1. Moreover, $\tilde{M}=[I|\tilde{B}]$ represents $\mathscr{M}$ over $\mathbb{R}$ and every binary vector x such that $Mx \equiv 0 \pmod 2$ corresponds to some 0, ± 1-vector y such that $\tilde{M}y=0$, where y is obtained from x by replacing its 1's by ± 1's.

Decomposition Result

The following decomposition result was proved by Tseng and Truemper ([14], Theorem 4.3); see also ([12], Theorem 1.3) and ([13], Chap. 13) for a detailed exposition.

THEOREM 2.3. *Let $\mathscr{M}$ be a matroid on the set $E \cup \{l\}$. Assume that $\mathscr{M}$ does not have any minor F_7^* using the element l. Then, one of the following holds*:

(i) *$\mathscr{M}$ has a 1-separation.*

(ii) *$\mathscr{M}$ is 2-connected and has a 2-separation.*

(iii) *$\mathscr{M}$ is a regular matroid.*

(iv) *$\mathscr{M}$ is the Fano matroid F_7.*

(v) *$\mathscr{M}$ is 3-connected and has a 3-separation $(E_1, E_2 \cup \{l\})$ such that (E_1, E_2) is a strict 2-separation of $\mathscr{M}/l$.*

Remark 2.4. Theorem 2.3 differs from Theorem 1.3 of [12] in the statement (v). However, the above formulation of (v) follows from Theorems 1.3 and 2.1 from [12] (the latter theorem states that the triple $\{y, z, l\}$ forms a circuit of $\mathscr{M}/(X_1 - x)\backslash(Y_1 - \{y, z\}))$ and from the above Proposition 2.2.

We will use this decomposition result in the following form.

THEOREM 2.5. *Let $\mathscr{M}$ be a binary matroid on the set $E \cup \{l\}$. Assume that $\mathscr{M}$ does not have any minor F_7^* using the element l and that $\mathscr{M}$ does not have any minor F_7^+ using the element l as a series element. Assume also that l is neither a loop nor a coloop of $\mathscr{M}$. Then, one of the following holds*:

(a) *$\mathscr{M}/l$ has a 1-separation.*

(b) *$\mathscr{M}/l$ has a strict 2-separation.*

(c) *$\mathscr{M}$ is regular.*

Proof. We apply Theorem 2.3. The statement (iii) coincides with (c). Moreover, (b) applies in cases (iv) and (v). In case (i), if $(E_1, E_2 \cup \{l\})$ is a 1-separation of $\mathscr{M}$, then (E_1, E_2) is a 1-separation of $\mathscr{M}/l$ since l is not a (co)loop of $\mathscr{M}$; hence, (a) applies. We suppose finally that we are in the case (ii), i.e., $(E_1, E_2 \cup \{l\})$ is a strict 2-separation of $\mathcal{M}$. If $r_{\mathcal{M}}(E_1) = r_{\mathcal{M}/l}(E_1) + 1$, then (E_1, E_2) is a 1-separation of $\mathcal{M}/l$ and, thus, (a) applies. Otherwise, $r_{\mathcal{M}}(E_1) = r_{\mathcal{M}/l}(E_1)$, implying that $r_{\mathcal{M}/l}(E_1) + r_{\mathcal{M}/l}(E_2) = r_{\mathcal{M}/l}(E) + 1$. Hence, in order to show that (b) applies, we need only to check that $|E_2| \geqslant 2$. Suppose, for contradiction, that $|E_2| = 1$, i.e., $E_2 = \{l'\}$. We deduce that $\{l, l'\}$ is a cocircuit of $\mathcal{M}$. Therefore, $\mathcal{M}$ can be seen as the series-extension of $\mathcal{M}/l$ obtained by adding l in series with l'. If $\mathcal{M}/l$ is regular, then $\mathcal{M}$ is regular too and, thus, (c) applies. Hence, we can suppose that $\mathcal{M}/l$ is 2-connected and not regular. It follows from [9] that $\mathcal{M}/l$ has a minor F_7 or F_7^* using l'. It is easy to see that, if $\mathcal{M}/l$ has a minor F_7^* using l', then $\mathcal{M}$ has a minor F_7^* using l and, if $\mathcal{M}/l$ has a minor F_7 using l', then $\mathcal{M}$ has a minor F_7^+ using l as a series element. We obtain a contradiction in both cases. ∎

Remark 2.6. One can check that under the conditions of Theorem 2.5 (i.e., $\mathcal{M}$ is a binary matroid having no minor F_7^* using l, no minor F_7^+ using l as a series element, and l is not a (co)loop of $\mathcal{M}$) $\mathcal{M}/l$ is regular or $\mathcal{M}$ has a 1-separation.

Signed Circuits

Let $\mathcal{M}$ be a binary matroid on $E \cup \{l\}$ and let $\mathcal{L}$ denote the l-port of $\mathcal{M}$. A convenient way to refer to the members of $\mathcal{L}$ is in terms of odd circuits of $\mathcal{M}/l$ with respect to some signing. Given a set $\Sigma \subseteq E + l$, a subset $A \subseteq E$ is called *Σ-even* (resp. *Σ-odd*) if $|A \cap \Sigma|$ is even (reps. odd). The following is easy to check.

PROPOSITION 2.7. *Let Σ be a cocircuit of $\mathcal{M}$ such that $l \in \Sigma$ and let C be a subset of E. Then, $C \in \mathcal{L}$ if and only if C is a Σ-odd circuit of $\mathcal{M}/l$.*

3. Q_6, Q_7, AND REGULAR CASE

In this section we show the following results:

- It is sufficient to work with fully fractional vertices, see Proposition 3.1.
- Box $\frac{1}{d}$-integrality is preserved under minors, see Proposition 3.2.
- Q_6, the port of F_7^*, is not box $\frac{1}{d}$-integral for any integer $d \geqslant 2$, see Proposition 3.3.
- Q_7, the port of the series-extension of F_7 with respect to a series element, is not box $\frac{1}{d}$-integral for any integer $d \geqslant 2$, see Proposition 3.4.
- Any port of a regular matroid is box $\frac{1}{d}$-integral for each integer $d \geqslant 1$, see Theorem 3.5.

The following result is easy to check.

PROPOSITION 3.1. *Let* $f \in E$, $I \subseteq E - f$, $a \in (\frac{1}{d}\mathbb{Z})^I$ *and* $x \in \mathbb{R}^{E-f}$. *Then,*

(i) *x belongs to (resp. is a vertex of) $Q(\mathscr{L}/f, a)$ if and only if $(x, 0)$ belongs to (resp. is a vertex of) $Q(\mathscr{L}, (a, 0))$.*

(ii) *x belongs to (resp. is a vertex of) $Q(\mathscr{L}\backslash f, a)$ if and only if $(x, 1)$ belongs to (resp. is a vertex of) $Q(\mathscr{L}, (a, 1))$.*

As an immediate consequence, we have that

PROPOSITION 3.2. *Every minor of a box $\frac{1}{d}$-integral clutter is box $\frac{1}{d}$-integral.*

PROPOSITION 3.3. *The clutter Q_6 is not box $\frac{1}{d}$-integral, for any integer $d \geqslant 2$.*

Proof. Consider the vector $u \in \mathbb{R}^6$ defined by $u_1 = 1 - \frac{1}{d}$, $u_2 = u_6 = \frac{1}{d}$, $u_3 = u_5 = \frac{1}{2d}$, $u_4 = 1 - \frac{3}{2d}$. Set $a_1 = 1 - \frac{1}{d}$, $a_2 = a_6 = \frac{1}{d}$. Then, u belongs to the polyhedron $Q(Q_6, a)$. In fact, it is a vertex of that polyhedron, since it satisfies the following six linearly independent equalities: $u_1 + u_3 + u_5 = 1$, $u_2 + u_3 + u_4 = 1$, $u_4 + u_5 + u_6 = 1$, $u_1 = a_1$, $u_2 = a_2$, and $u_6 = a_6$. ∎

PROPOSITION 3.4. *The clutter Q_7 is not box $\frac{1}{d}$-integral, for any integer $d \geqslant 2$.*

Proof. Consider the vector $u \in \mathbb{R}^7$ defined by $u_1 = u_3 = u_5 = \frac{1}{2d}$, $u_2 = u_4 = u_6 = \frac{1}{d}$, and $u_7 = 1 - \frac{3}{2d}$. Set $a_2 = a_4 = a_6 = \frac{1}{d}$. Then, u belongs to the polyhedron $Q(Q_7, a)$. In fact, it is a vertex of that polyhedron, since it satisfies the following seven linearly independent equalities: $u_1 + u_4 + u_7 = 1$, $u_2 + u_5 + u_7 = 1$, $u_3 + u_6 + u_7 = 1$, $u_1 + u_3 + u_5 + u_7 = 1$, $u_2 = a_2$, $u_4 = a_4$, and $u_6 = a_6$. ∎

THEOREM 3.5. *Let $\mathscr{M}$ be the port of a regular matroid. Then, $\mathscr{M}$ is box $\frac{1}{d}$-integral for each integer $d \geqslant 1$.*

Proof. Let $\mathscr{M}$ be a regular matroid on $E \cup \{l\}$ and let $\mathscr{L}$ be its l-port. If l is a loop then $\mathscr{L} = \{\emptyset\}$, so $\mathscr{L}$ is box $\frac{1}{d}$-integral. We suppose now that l is not a loop. Since $\mathscr{M}$ is regular, we can find a totally unimodular matrix M which represents $\mathscr{M}$ over $\mathbb{R}$ and is of the form shown in Fig. 4. We can suppose that the matrix A has full row rank.
Moreover, each set $C \in \mathscr{L}$ corresponds to a vector $y_C \in \{0, 1, -1\}^E$ such that

$$\begin{cases} r^T y_C = 1 \\ A y_C = 0. \end{cases}$$

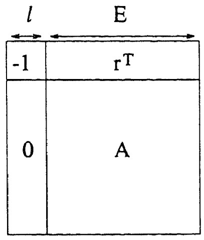

FIGURE 4

Each such y_C can be written as $y_C = y_C^1 - y_C^2$, where $y_C^1, y_C^2 \in \{0, 1\}^E$ and their supports $\{e \in E \mid (y_C^1)_e = 1\}$, $\{e \in E \mid (y_C^2)_e = 1\}$ partition the set C.

We define the polyhedron $\mathscr{K}$ consisting of the vectors $(y_1, y_2) \in \mathbb{R}^E \times \mathbb{R}^E$ satisfying

$$\begin{cases} r^T y_1 - r^T y_2 = 1, \\ A y_1 - A y_2 = 0, \\ y_1, y_2 \geqslant 0. \end{cases}$$

Clearly, $(y_C^1, y_C^2) \in \mathscr{K}$ for each $C \in \mathscr{L}$. We state a preliminary result.

Claim 3.6. Let $u \in \mathbb{R}_+^E$. Then,

(i) $\min(u(C) \mid C \in \mathscr{L}) = \min(u^T y_1 + u^T y_2 \mid (y_1, y_2) \in \mathscr{K})$.

(ii) $u(C) \geqslant 1$ for all $C \in \mathscr{L}$ if and only if the system

$$\begin{cases} r^T + \pi^T A \leqslant u^T \\ -r^T - \pi^T A \leqslant u^T \end{cases}$$

(in the variable π) is feasible.

Proof. (i) The first minimum is greater or equal to the second one, since each $C \in \mathscr{L}$ corresponds to a pair $(y_C^1, y_C^2) \in \mathscr{K}$ such that $u(C) = u^T y_C^1 + u^T y_C^2$. Let (y_1, y_2) be a vertex of $\mathscr{K}$ at which the second minimum is attained. Clearly, the supports of y_1, y_2 are disjoint. Since the matrix M is totally unimodular, we deduce that $y_1, y_2 \in \{0, 1\}^E$. Set $C = \{e \in E \mid (y_1)_e = 1 \text{ or } (y_2)_e = 1\}$. Then, $C \in \mathscr{L}$ and C corresponds to the vector $y_C = y_1 - y_2$ with $u^T y_1 + u^T y_2 = u(C)$. This shows that the second minimum is greater or equal to the first one.

(ii) Observe that the system $\left\{\begin{smallmatrix} r^T + \pi^T A \leqslant u^T \\ -r^T - \pi^T A \leqslant u^T \end{smallmatrix}\right.$ is feasible if and only if

$$\max(\rho \mid \rho r^T + \pi^T A \leqslant u^T,\ -\rho r^T - \pi^T A \leqslant u^T) \geqslant 1.$$

Moreover, by linear programming duality and Claim 3.6(i), we obtain:

$$\max(\rho \mid \rho r^T + \pi^T A \leqslant u^T,\ -\rho r^T - \pi^T A \leqslant u^T)$$
$$= \min(u^T y_1 + u^T y_2 \mid (y_1, y_2) \in \mathscr{K})$$
$$= \min(u(C) \mid C \in \mathscr{L}). \quad \blacksquare$$

Let I be a subset of E and let $a \in (\frac{1}{d}\mathbb{Z})^I$. Let $\tilde{Q}(\mathscr{L}, a)$ denote the polyhedron consisting of the vectors $(\pi, u) \in \mathbb{R}^m \times \mathbb{R}^E$ (m denoting the number of rows of the matrix A) satisfying

$$\begin{cases} \pi^T A - u^T \leqslant -r^T, \\ -\pi^T A - u^T \leqslant r^T, \\ u_e = a_e \qquad \text{for} \quad e \in I. \end{cases}$$

Note that $\tilde{Q}(\mathscr{L}, a)$ has vertices as the matrix A has full row rank. By Claim 3.6(ii), $Q(\mathscr{L}, a)$ is the projection of $\tilde{Q}(\mathscr{L}, a)$ on the subspace $\mathbb{R}^E$.

Let u_0 be a vertex of $Q(\mathscr{L}, a)$. By Proposition 3.1, we can suppose that all components of u_0 are positive. Moreover, u_0 is the projection of a vertex (π_0, u_0) of $\tilde{Q}(\mathscr{L}, a)$. Since $\tilde{Q}(\mathscr{L}, a)$ is invariant under the multiplication of some columns of the matrix

$$\begin{bmatrix} r^T \\ \hline A \end{bmatrix}$$

by -1, we may assume that $\pi_0^T A + r^T \geqslant 0$ and, thus, that $-\pi_0^T A - u_0^T < r^T$. Therefore, (π_0, u_0) is a vertex of the polyhedron

$$\{(\pi, u) \mid \pi^T A - u^T \leqslant -r^T, u_e = a_e \text{ for } e \in I\}.$$

As the matrix defining this polyhedron is totally unimodular, we deduce that (π_0, u_0) is $\frac{1}{d}$-integral. This shows that u_0 is $\frac{1}{d}$-integral. (Note that the constraint matrix for $\tilde{Q}(\mathscr{L}, a)$ is *not* totally unimodular.) $\blacksquare$

4. Proof of the Main Result

Let $\mathscr{M}$ be a binary matroid on $E \cup \{l\}$ and let $\mathscr{L}$ be the l-port of $\mathscr{M}$, i.e., $\mathscr{L} = \{C \subseteq E \mid C + l \text{ is a circuit of } \mathscr{M}\}$. Let $d \geqslant 1$ be an integer. We assume that $\mathscr{L}$ does not have Q_6 or Q_7 as a minor. Hence, $\mathscr{M}$ does not have F_7^* as a minor using l and $\mathscr{M}$ does not have F_7^+ as a minor using l as a series element.

Our goal is to show that $\mathscr{L}$ is box $\frac{1}{d}$-integral. The proof is by induction on $|E| \geqslant 0$ and the main tool we use is Theorem 2.5.

The result holds for $|E| = 0$. Indeed, then l is either a loop, yielding $\mathscr{L} = \{\emptyset\}$, or a coloop, yielding $\mathscr{L} = \emptyset$. In both cases, $\mathscr{L}$ is box $\frac{1}{d}$-integral.

We assume that the result holds for every groundset with less than $|E|$ elements, i.e., that every binary clutter without Q_6 or Q_7 minor on a set with less than $|E|$ elements is box $\frac{1}{d}$-integral.

We can suppose that l is neither a loop nor a coloop of $\mathscr{M}$. We know from Theorem 3.5 that $\mathscr{L}$ is box $\frac{1}{d}$-integral if $\mathscr{M}$ is regular. From Theorem 2.5, we can assume that $\mathscr{M}/l$ has a 1-separation or a strict 2-separation.

PROPOSITION 4.1. *If $\mathscr{M}/l$ has a 1-separation, then $\mathscr{L}$ is box $\frac{1}{d}$-integral.*

Proof. Let (E_1, E_2) be a 1-separation of $\mathscr{M}/l$. Let $\mathscr{L}_1$ (resp. $\mathscr{L}_2$) denote the l-port of the matroid $\mathscr{M}\backslash E_2$ (resp. $\mathscr{M}\backslash E_1$). Clearly, $\mathscr{L}_1 \cup \mathscr{L}_2 \subseteq \mathscr{L}$; in fact, $\mathscr{L}_1$ and $\mathscr{L}_2$ partition $\mathscr{L}$. By the induction assumption, $\mathscr{L}_1$ and $\mathscr{L}_2$ are box $\frac{1}{d}$-integral.

Given $I \subseteq E$ and $a \in (\frac{1}{d}\mathbb{Z})^I$, set $a_i = (a_e)_{e \in I \cap E_i}$, for $i = 1, 2$. Then, $Q(\mathscr{L}, a)$ is the cartesian product of $Q(\mathscr{L}_1, a_1)$ and $Q(\mathscr{L}_2, a_2)$, which implies that all its vertices are $\frac{1}{d}$-integral. ∎

From now on we assume that $\mathscr{M}/l$ is 2-connected and admits a 2-separation (E_1, E_2). Let I be a subset of E, let $a \in (\frac{1}{d}\mathbb{Z})^I$, and let u be a vertex of $Q(\mathscr{L}, a)$. Our goal is to show that u is $\frac{1}{d}$-integral. From Proposition 3.1 and the induction hypothesis, we can suppose that $u_e \neq 0, 1$, for all $e \in E$. Call an inequality *tight* for u if it is satisfied at equality by u.

The inequalities defining $Q(\mathscr{L}, a)$ are of three types:

Type I: $x_e = a_e$ for $e \in I$.

Type II: $x(C) \geqslant 1$ for each noncrossing $C \in \mathscr{L}$ (i.e., $C \subseteq E_i$ for $i \in \{1, 2\}$).

Type III: $x(C) \geqslant 1$ for each crossing $C \in \mathscr{L}$.

The case when no inequality of type III is tight for u is easy:

PROPOSITION 4.2. *If $u(C) > 1$ for each crossing $C \in \mathscr{L}$, then u is $\frac{1}{d}$-integral.*

Proof. The proof is analogous to that of Proposition 4.1. ∎

We now suppose that there exists some crossing $C \in \mathscr{L}$ with $u(C) = 1$.

DEFINITION 4.3. We call *path* every set of the form $C \cap E_i$ where $i \in \{1, 2\}$ and $C \in \mathscr{L}$ is crossing.

Let Σ be a cocircuit of $\mathcal{M}$ which contains l. Set

$$u_o = \min(u(P) \mid P \text{ is a path with } |P \cap \Sigma| \text{ odd}),$$

$$u_e = \min(u(P) \mid P \text{ is a path with } |P \cap \Sigma| \text{ even}).$$

Both u_o, u_e are well defined.

PROPOSITION 4.4. *We have that* $u_o + u_e = 1$. *Moreover, for each tight crossing* $C \in \mathcal{L}$ *with, say,* $C \cap E_1$ Σ*-odd and* $C \cap E_2$ Σ*-even,* $u(C \cap E_1) = u_o$ *and* $u(C \cap E_2) = u_e$.

Proof. Take $C \in \mathcal{L}$ crossing and tight. Then, $1 = u(C) = u(C \cap E_1) + u(C \cap E_2) \geqslant u_o + u_e$. Conversely, suppose that $u_o = u(C \cap E_i)$ and $u_e = u(C' \cap E_j)$, where $C, C' \in \mathcal{L}$ are crossing with $C \cap E_i$ Σ-odd, $C' \cap E_j$ Σ-even and $i, j \in \{1, 2\}$. From Proposition 2.1(ii), $C'' = (C \cap E_i) \triangle (C' \cap E_j)$ is a cycle of $\mathcal{M}/l$. Hence, $C'' = \bigcup_h C_h$, where C_h are pairwise disjoint circuits of $\mathcal{M}/l$. Since C'' is Σ-odd, at least one of the C_h's is Σ-odd, i.e., belongs to $\mathcal{L}$. This implies that $u(C'') = \sum_h u(C_h) \geqslant 1$. Therefore, $u_o + u_e \geqslant 1$. Hence, we have the equality $u_o + u_e = 1$. The last part of the proposition follows immediately. ∎

Let $\mathcal{B}$ be a base of equalities for u, i.e., $\mathcal{B}$ is a maximal set of linearly independent inequalities chosen among the inequalities defining $Q(\mathcal{L}, a)$ that are satisfied at equality by u. Let $\mathcal{B}_i$ denote the subset of $\mathcal{B}$ consisting of the inequalities which are supported by E_i, for $i = 1, 2$. Hence, $\mathcal{B}_1 \cup \mathcal{B}_2$ consists of inequalities of Type I or II and $\mathcal{B} - \mathcal{B}_1 \cup \mathcal{B}_2$ of inequalities of Type III. We can partition $\mathcal{B} - \mathcal{B}_1 \cup \mathcal{B}_2$ as $\mathcal{B}_3 \cup \mathcal{B}_4$, where $\mathcal{B}_3$ consists of inequalities $x(C) \geqslant 1$ for $C \in \mathcal{L}$ crossing with $C \cap E_1$ Σ-odd, $C \cap E_2$ Σ-even, and $\mathcal{B}_4$ of such inequalities with $C \in \mathcal{L}$ crossing, $C \cap E_1$ Σ-even and $C \cap E_2$ Σ-odd.

PROPOSITION 4.5. *There exists a base* $\mathcal{B}$ *of equalities for* u *for which* $\mathcal{B}_3 = \varnothing$ *or* $\mathcal{B}_4 = \varnothing$.

Proof. Let $\mathcal{B}$ be a base of equalities for u for which $|\mathcal{B}_1 \cup \mathcal{B}_2|$ is maximum. Suppose, for contradiction, that $\mathcal{B}_3 \neq \varnothing$ and $\mathcal{B}_4 \neq \varnothing$. Let $C, C' \in \mathcal{L}$ be crossing and yielding equalities of $\mathcal{B}$ with $C \cap E_1$, $C' \cap E_2$ Σ-even and $C \cap E_2$, $C' \cap E_1$ Σ-odd. By Proposition 2.1(ii), $D_i := (C \cap E_i) \triangle (C' \cap E_i)$ is a cycle of $\mathcal{M}/l$. Moreover, D_i is Σ-odd by construction. Hence, $D_i = \bigcup_h C_h$ where the C_h's are pairwise disjoint circuits of $\mathcal{M}/l$ and at least one of them is Σ-odd. Using Proposition 4.4, we obtain that $1 = u_e + u_o \geqslant u(D_i) \geqslant 1$ which implies that $C \cap C' = \varnothing$ and that D_1 and D_2 are (noncrossing) circuits of $\mathcal{M}/l$, each yielding a tight equality for u. The base $\mathscr{B}$ cannot contain both equations $x(D_1)=1$ and $x(D_2)=1$ since $C \cup C' = D_1 \cup D_2$. If $\mathscr{B}$ contains $x(D_1)=1$ but not $x(D_2)=1$, then, by replacing the equation $x(C')=1$ by the equation $x(D_2)=1$, we obtain a new base $\mathscr{B}'$ (this follows from the fact that $\mathscr{B}$ is a base and the relation $x(C)+x(C')=x(D_1)+x(D_2)$). As $\mathscr{B}'$ satisfies: $|\mathscr{B}'_1 \cup \mathscr{B}'_2| > |\mathscr{B}_1 \cup \mathscr{B}_2|$, we have a contradiction with the choice of $\mathscr{B}$. Therefore, $\mathscr{B}$ contains none of the equations $x(D_1)=1$, $x(D_2)=1$. At least one of them can be added to $\mathscr{B}$ after deleting the equation $x(C')=1$, still preserving linear independence. Again we obtain a contradiction with the maximality of $|\mathscr{B}_1 \cup \mathscr{B}_2|$. ∎

By symmetry, we can suppose that we have a base $\mathscr{B}$ of equalities for u with $\mathscr{B}_4 = \emptyset$, $\mathscr{B}_3 \neq \emptyset$. (If both $\mathscr{B}_3$ and $\mathscr{B}_4$ are empty, we can conclude in the same way as in Proposition 4.2.) In matrix form, the system $\mathscr{B}$ can be written as $Px=\beta$, where β is the vector consisting of the right hand sides of the inequalities and P is the nonsingular matrix shown in Fig. 5.

Hence, there exists a tight equality $u(C^*)=1$ where $C^* \in \mathscr{L}$ is crossing, $C^* \cap E_1$ is Σ-odd and $C^* \cap E_2$ is Σ-even. We can find two elements $e_1 \in C^* \cap E_2$, $e_2 \in C^* \cap E_1$ with $e_1 \notin \Sigma$ and $e_2 \in \Sigma$ (after eventually changing the cocircuit Σ). (Indeed, let $e_2 \in C^* \cap E_1$, $e_1 \in C^* \cap E_2$ and let X be a base of $\mathscr{M}$ containing $(C^* - e_2) \cup \{l\}$. Let Σ' denote the fundamental cocircuit of l in the base X; then, $e_2 \in \Sigma'$ since $C^* + l$ is the fundamental circuit of e_2 in the base X, and $e_1 \notin \Sigma'$ since $e_1 \in X$. Hence, it suffices to replace Σ by Σ').

Set $\mathscr{M}_1 = \mathscr{M}/((C^* \cap E_2) - e_1)\backslash(E_2 - C^*)$ and $\mathscr{M}_2 = \mathscr{M}/((C^* \cap E_1) - e_2)\backslash(E_1 - C^*)$, defined, respectively, on the sets $E_1 \cup \{e_1, l\}$ and $E_2 \cup \{e_2, l\}$. (Note that $\mathscr{M}_1$ coincides with $\mathscr{M}/(X_2 - e_1)\backslash Y_2$ and $\mathscr{M}_2$ coincides with $\mathscr{M}/X_1\backslash(Y_1 - e_2)$, where $X_i = X \cap E_i$, $Y_i = E_i - X_i$ for $i = 1, 2$. Also, $\mathscr{M}/l$ is the 2-sum of $\mathscr{M}_1/l$ and $\mathscr{M}_2/l$. Recall Section 2.)

Let $\mathscr{L}_i$ denote the l-port of $\mathscr{M}_i$. By the induction assumption, $\mathscr{L}_i$ is box $\frac{1}{d}$-integral, for $i = 1, 2$.

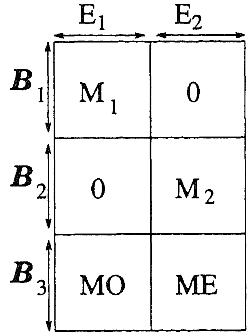

FIGURE 5

Let u_i denote the projection of u on $\mathbb{R}^{E_i}$ and set $a_i = (a_e)_{e \in I \cap E_i}$, for $i = 1, 2$. We define $u_i^* \in \mathbb{R}^{E_i + e_i}$ by

$$\begin{cases} u_i^*(e) = u_i(e) & \text{for} \quad e \in E_i, \quad i = 1, 2, \\ u_1^*(e_1) = u_e, \\ u_2^*(e_2) = u_o. \end{cases}$$

Proposition 4.6. *$u_i^* \in Q(\mathscr{L}_i, a_i)$, for $i = 1, 2$.*

Proof. We give the proof for $i = 1$, the case $i = 2$ is identical. Take $C \in \mathscr{L}_1$. By Proposition 2.1(i), either $C \in \mathscr{L}$ and, thus, $u_1^*(C) = u(C) \geqslant 1$, or $C = C' \cap E_1 + e_1$ for some crossing circuit C' of $\mathscr{M}/l$. Then, $C' \cap E_1$ is Σ-odd, since C is Σ-odd and $e_1 \notin \Sigma$. By Proposition 2.1(ii), $(C' \cap E_1) \triangle (C^* \cap E_2)$ is a cycle of $\mathscr{M}/l$ and it is Σ-odd since $C^* \cap E_2$ is Σ-even. Hence, $u(C' \cap E_1) + u(C^* \cap E_2) \geqslant 1$ which, together with $u(C^* \cap E_2) = u_e$, implies that $u(C' \cap E_1) \geqslant 1 - u_e = u_o$. Therefore, $u_1^*(C) = u(C' \cap E_i) + u_e \geqslant u_o + u_e = 1$. ∎

We construct the set $\mathscr{B}^{(i)}$ of equalities for u_i^* consisting of

- the equalities of $\mathscr{B}_i$,
- the equalities $x((C \cap E_i) + e_i) = 1$, one for each equality $x(C) = 1$ of $\mathscr{B}_3$.

All equalities of $\mathscr{B}^{(i)}$ arise from those defining $Q(\mathscr{L}_i, a_i)$. Indeed, by Proposition 2.1, if $C \in \mathscr{L}$ with $C \subseteq E_i$, then $C \in \mathscr{L}_i$ and, if $C \in \mathscr{L}$ is crossing, then $(C \cap E_i) + e_i \in \mathscr{L}_i$, for $i = 1, 2$.

Proposition 4.7. *The set $\mathscr{B}^{(i)}$ has rank $|E_i| + 1$ for at least one index $i \in \{1, 2\}$.*

Proof. We show that one of the two matrices in Figs. 6 and 7 has full rank $|E_i| + 1$.

As the matrix P of Fig. 5 has full rank $|E_1| + |E_2|$, it follows easily that the matrix displayed in Fig. 9 has full rank $|E_1| + |E_2| + 2$. This implies that

M_1	0
MO	1

Figure 6

0	M_2
1	ME

FIGURE 7

M_1	0	0	0
MO	1	0	0
0	0	0	M_2
0	0	1	ME
0	1	1	0

FIGURE 8

M_1	0	0	0
0	0	0	M_2
MO	0	0	ME
-MO	0	1	0
0	1	0	-ME

FIGURE 9

the matrix shown in Fig. 8 has also full rank $|E_1| + |E_2| + 2$, as it can be obtained by row and column operations from the matrix in Fig. 9. ∎

By symmetry, we can suppose that $\mathscr{B}^{(1)}$ has full rank. This implies that u_1^* is a vertex of $Q(\mathscr{L}_1, a_1)$ and, thus, u_1^* is $\frac{1}{d}$-integral, since $\mathscr{L}_1$ is box $\frac{1}{d}$-integral. In particular, u_e is $\frac{1}{d}$-integral, implying that $u_o = 1 - u_e$ is $\frac{1}{d}$-integral. If we introduce the constraint $x(e_2) = u_o$, then u_2^* becomes a vertex of the polytope $Q(\mathscr{L}_2, a_2) \cap \{x \mid x(e_2) = u_o\}$ and, thus, u_2^* is $\frac{1}{d}$-integral.

This shows that u is $\frac{1}{d}$-integral and concludes the proof. ∎

5. Applications for Graphs

A *signed graph* is a pair (G, Σ), where $G = (V, E)$ is a graph and Σ is a subset of the edge set E of G. The edges in Σ are called *odd* and the other edges *even*. An *odd circuit* C in (G, Σ) is a circuit C of G such that $|C \cap \Sigma|$ is odd. If $\delta(U)$ is a cut in G, then the two signed graphs (G, Σ) and $(G, \Sigma \triangle \delta(U))$ have the same collection of odd circuits. The operation $\Sigma \to \Sigma \triangle \delta(U)$ is called *resigning* (by the cut $\delta(U)$). We say that (G, Σ) *reduces* to (G', Σ') if (G', Σ') can be obtained from (G, Σ) by a sequence of the following operations:

- deleting an edge of G (and Σ),
- contradicting an even edge of G,
- resigning.

The collection of odd circuits of a signed graph is a binary clutter. Indeed, given a signed graph (G, Σ), let $\mathscr{S}(G, \Sigma)$ denote the binary matroid on $\{l\} \cup E$ represented over $GF(2)$ by the matrix

$$\left[\begin{array}{c|c} 1 & \sigma \\ \hline 0 & M_G \end{array}\right]$$

where M_G is the node-edge incidence matrix of G and σ is the incidence vector of the set Σ. Clearly, the l-port of $\mathscr{S}(G, \Sigma)$ coincides with the family of odd circuits of (G, Σ). In particular, the collection of odd circuits of the signed graph $(K_4, E(K_4))$, i.e., K_4 with all edges odd, is the clutter Q_6, i.e. $\mathscr{S}(K_4, E(K_4))$ is F_7^*. One can check that (G, Σ) does not reduce to $(K_4, E(K_4))$ if and only if $\mathscr{S}(G, \Sigma)$ does not have an F_7^* minor using the element l. Moreover, $\mathscr{S}(G, \Sigma)$ does not have any minor F_7^+ using l as a series element, otherwise F_7 would be a minor of the graphic matroid $\mathscr{M}(G) = \mathscr{S}(G, \Sigma)/l$. (See [5] for details.)

The following result is an immediate application of Theorem 1.2.

THEOREM 5.1. *Let (G, Σ) be a signed graph and let $\mathscr{L}$ denote its collection of odd circuits. The following assertions are equivalent.*

(i) *(G, Σ) does not reduce to $(K_4, E(K_4))$.*

(ii) *$\mathscr{L}$ is box $\frac{1}{d}$-integral for any integer $d \geqslant 1$.*

(iii) *$\mathscr{L}$ is box $\frac{1}{d}$-integral for some integer $d \geqslant 2$.*

Given a graph $G = (V, E)$, we consider the polytope

$$R(G) = \{x \in \mathbb{R}^E \mid x(F) - x(C-F) \leqslant |F| - 1 \ (C \text{ circuit of } G, F \subseteq C, |F| \text{ odd}),$$
$$0 \leqslant x_e \leqslant 1 \ (e \in E)\}.$$

The polytope $R(G)$ is a relaxation of the cut polytope $P(G)$ (defined as the convex hull of the incidence vectors of the cuts of G). In general, $R(G)$ has fractional vertices. In fact, the 0, 1-vertices of $R(G)$ are the incidence vectors of the cuts of G, and $R(G)$ has only integral vertices, i.e. $R(G) = P(G)$, if and only if G does not have K_5 as a minor [2]. The fractional vertices of $R(G)$ have been studied in [6], [7].

The case $d = 3$ of the following Theorem 5.2. was proved in [7]. We will show how Theorem 5.2. follows from Theorem 5.1.

THEOREM 5.2. *Let $G = (V, E)$ be a graph. The following assertions are equivalent.*

(i) *G is series parallel, i.e., G does not have K_4 as a minor.*

(ii) *For each $I \subseteq E$ and $a \in (\frac{1}{d}\mathbb{Z})^I$, all the vertices of the polytope $R(G) \cap \{x \mid x_e = a_e \text{ for } e \in I\}$ are $\frac{1}{d}$-integral, for any integer $d \geqslant 1$.*

(iii) *For each $I \subseteq E$ and $a \in (\frac{1}{d}\mathbb{Z})^I$, all the vertices of the polytope $R(G) \cap \{x \mid x_e = a_e \text{ for } e \in I\}$ are $\frac{1}{d}$-integral, for some integer $d \geqslant 2$.*

Proof. Let $G' = (V, E \cup E')$ denote the graph obtained from G by adding an edge e' in parallel with each edge e of G. We consider the signed graph (G', E'), so the edges of E are even and those of E' are odd. It is easy to see that G is series parallel if and only if (G', E') does not reduce to $(K_4, E(K_4))$. Let $\mathscr{L}'$ denote the collection of odd circuits of (G', E'). From Theorem 5.1, $\mathscr{L}'$ is box $\frac{1}{d}$-integral if G is series parallel. For $x \in \mathbb{R}^E$, define $x' \in \mathbb{R}^{E'}$ by $x'_{e'} = 1 - x_e$ for $e \in E$ and, for $a \in (\frac{1}{d}\mathbb{Z})^I$ with $I \subseteq E$, set $a'_{e'} = 1 - a_e$ for $e \in I$.

Observe that $R(G) \cap \{x \mid x_e = a_e \text{ for } e \in I\} = \{x \mid (x, x') \in Q(\mathscr{L}', (a, a'))\}$. As $\{e, e'\} \in \mathscr{L}'$ for each $e \in E$, $Q(\mathscr{L}', (a, a')) \cap \{(x, y) \in \mathbb{R}^E \times \mathbb{R}^{E'} \mid y_{e'} = 1 - x_e$ for $e \in E\}$ is a face of $Q(\mathscr{L}', (a, a'))$. Therefore, $R(G) \cap \{x \mid x_e = a_e$ for $e \in I\}$ is the projection of a face of $Q(\mathscr{L}', (a, a'))$. Hence, all its vertices are $\frac{1}{d}$-integral if G is series parallel. This proves (i) $\Rightarrow$ (ii).

It is easy to check that (iii) is closed under graph minors. Moreover, K_4 does not have the property (iii). Indeed, consider K_4 with its edges labeled 1, 2, 3, 4, 5, 6 in such a way that the triangles of K_4 are $\{1, 2, 6\}$, $\{1, 3, 5\}$, $\{2, 3, 4\}$, $\{4, 5, 6\}$ (i.e., the members of Q_6). Set $x_2 = x_4 = x_6 = \frac{1}{d}$ and $x_1 = x_3 = x_5 = \frac{1}{2d}$. Then, x is a non $\frac{1}{d}$-integral vertex of the polytope $R(K_4) \cap \{x \mid x_i = \frac{1}{d}$ for $i = 2, 4, 6\}$. This shows (iii) $\Rightarrow$ (i). ∎

More generally, given a binary matroid $\mathcal{M}$ on a set E, consider the polytope $R(\mathcal{M})$ in $\mathbb{R}^E$ defined by the inequalities $0 \leqslant x_e \leqslant 1$ for $e \in E$, and $x(F) - x(C-F) \leqslant |F| - 1$ for $F \subseteq C$ with $|F|$ odd and C circuit of $\mathcal{M}$. Hence, $R(\mathcal{M})$ coincides with $R(G)$ when $\mathcal{M}$ is the graphic matroid $\mathcal{M}(G)$ of G. The 0, 1-vertices of $R(\mathcal{M})$ are the incidence vectors of the cocycles of $\mathcal{M}$. The matroids $\mathcal{M}$ for which all vertices of $R(\mathcal{M})$ are integral have been characterized in [1] using a result of [11]. A natural question to ask is what are the matroids $\mathcal{M}$ for which $R(\mathcal{M})$ is box $\frac{1}{d}$-integral. Actually, this class is not larger than in the graphic case. To see this, observe that $F_7^*/l = \mathcal{M}(K_4)$ and that $F_7^+/l = F_7$ has an $\mathcal{M}(K_4)$ minor. On the other hand, a binary matroid $\mathcal{M}$ has no $\mathcal{M}(K_4)$ minor if and only if $\mathcal{M}$ is the graphic matroid of a series parallel graph. The latter follows easily from Tutte's forbidden minor characterization of graphic matroids ([16]).

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